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SOME DEFINITE INTEGRALS INVOLVING HYPERGEOMETRIC AND GAMMA FUNCTIONS

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Abstract: In this article, the authors make use of the Gamma function as well as the hypergeometric functions in order to investigate and develop five definite integrals involving the elliptic integrals. The numerical approximation of these definite integrals and the corresponding hypergeometric functions are also presented. The results derived in this article are believed to be new and extend and unify those that are available in the scientific literature.

Keywords and Phrases: Bessel Function, Meijer G-function.

2020 Mathematics Subject Classification: 33B50, 33D50, 33D60, 33D67.

1. Introduction

Recently Chaudhary gave a simple solution of some integrals given by Srinivasa Ramanujan [2]; and Chaudhary *et al.* studied about definite integrals [5], hyperge

-ometric functions [3, 4, 6, 7]. On the basis of current research progress, here we are discussing some definite integrals involving hypergeometric and gamma functions.

Bessel function of first kind of order n is defined as:

$$J_n(\eta) = \frac{(\frac{\eta}{2})^n}{\Gamma(n+1)}\ {}_0F_1\Big(-; n+1; -\frac{\eta^2}{4}\Big) \tag{1.1}$$

Seventeen Ramanujan's Series are[1, 8, 9]

$$R_4 \equiv \frac{4}{\pi} = 1 + \frac{7}{4}\Big(\frac{1}{2}\Big)^3 + \frac{13}{4^2}\Big(\frac{1.3}{2.4}\Big)^3 + \frac{19}{4^3}\Big(\frac{1.3.5}{2.4.6}\Big)^3 + \cdots = \sum_{n=0}^{\infty} \frac{(6n+1)(\frac{1}{2})_n}{4^n(n!)^3} \tag{1.2}$$

$$R_5 \equiv \frac{16}{\pi} = 5 + \frac{47}{64}\Big(\frac{1}{2}\Big)^3 + \frac{89}{64^2}\Big(\frac{1.3}{2.4}\Big)^3 + \frac{131}{64^3}\Big(\frac{1.3.5}{2.4.6}\Big)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(42n+5)\big(\frac{1}{2}\big)_n^3}{64^n(n!)^3} \tag{1.3}$$

$$R_6 \equiv \frac{32}{\pi} = (5\sqrt{5}-1) + \Big(\frac{47\sqrt{5}+29}{64}\Big)\Big(\frac{1}{2}\Big)^3\Big(\frac{\sqrt{5}-1}{2}\Big)^8$$

$$+\Big(\frac{89\sqrt{5}+59}{64^2}\Big)\Big(\frac{1.3}{2.4}\Big)^3\Big(\frac{\sqrt{5}-1}{2}\Big)^{16} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(42\sqrt{5}\ n + 5\sqrt{5} + 30n - 1)\big(\frac{1}{2}\big)_n^3}{64^n(n!)^3}\Big(\frac{\sqrt{5}-1}{2}\Big)^{8n} \tag{1.4}$$

$$R_7 \equiv \frac{27}{4\pi} = 2 + 17\ \frac{1}{2}\ \frac{1}{3}\ \frac{2}{3}\Big(\frac{2}{27}\Big) + 32\ \frac{1.3}{2.4}\ \frac{1.4}{3.6}\ \frac{2.5}{3.6}\ \Big(\frac{2}{27}\Big)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(15n+2)\big(\frac{1}{2}\big)_n\big(\frac{1}{3}\big)_n\big(\frac{2}{3}\big)_n}{(n!)^3}\Big(\frac{2}{27}\Big)^n \tag{1.5}$$

$$R_8 \equiv \frac{15\sqrt{3}}{2\pi} = 4 + 37\ \frac{1}{2}\ \frac{1}{3}\ \frac{2}{3}\ \Big(\frac{4}{125}\Big) + 70\ \frac{1.3}{2.4}\ \frac{1.4}{3.6}\ \frac{2.5}{3.6}\ \Big(\frac{4}{125}\Big)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(33n+4)\big(\frac{1}{2}\big)_n\big(\frac{1}{3}\big)_n\big(\frac{2}{3}\big)_n}{(n!)^3}\Big(\frac{4}{125}\Big)^n \tag{1.6}$$

$$R_9 \equiv \frac{5\sqrt{5}}{2\pi\sqrt{3}} = 1 + 12\ \frac{1}{2}\ \frac{1}{6}\ \frac{5}{6}\ \Big(\frac{4}{125}\Big) + 23\ \frac{1.3}{2.4}\ \frac{1.7}{6.12}\ \frac{5.11}{6.12}\ \Big(\frac{4}{125}\Big)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(11n+1)\big(\frac{1}{2}\big)_n\big(\frac{1}{6}\big)_n\big(\frac{5}{6}\big)_n}{(n!)^3}\Big(\frac{4}{125}\Big)^n \tag{1.7}$$

$$R_{10} \equiv \frac{85\sqrt{85}}{18\pi\sqrt{3}} = 8+141\,\frac{1}{2}\,\frac{1}{6}\,\frac{5}{6}\left(\frac{4}{85}\right)^3+274\,\frac{1.3}{2.4}\,\frac{1.7}{6.12}\,\frac{5.11}{6.12}\left(\frac{4}{85}\right)^6+\cdots$$

$$= \sum_{n=0}^{\infty} \frac{(133n+8)\left(\frac{1}{2}\right)_n\left(\frac{1}{6}\right)_n\left(\frac{5}{6}\right)_n}{(n!)^3}\left(\frac{4}{85}\right)^n \tag{1.8}$$

$$R_{11} \equiv \frac{4}{\pi} = \frac{3}{2} - \frac{23}{2^3}\,\frac{1}{2}\,\frac{1.3}{4^2} + \frac{43}{2^5}\,\frac{1.3}{2.4}\,\frac{1.3.5.7}{4^2.8^2} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(20n+3)\left(\frac{1}{2}\right)_n\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^3\;2^{2n+1}} \tag{1.9}$$

$$R_{12} \equiv \frac{4}{\pi\sqrt{3}} = \frac{3}{4} - \frac{31}{3.4^3}\,\frac{1}{2}\,\frac{1.3}{4^2} + \frac{59}{3^2.4^5}\,\frac{1.3}{2.4}\,\frac{1.3.5.7}{4^2.8^2} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(28n+3)\left(\frac{1}{2}\right)_n\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^3\;4^{n+1}} \tag{1.10}$$

$$R_{13} \equiv \frac{4}{\pi} = \frac{23}{18} - \frac{283}{18^3}\,\frac{1}{2}\,\frac{1.3}{4^2} + \frac{543}{18^5}\,\frac{1.3}{2.4}\,\frac{1.3.5.7}{4^2.8^2} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(260n+23)\left(\frac{1}{2}\right)_n\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^3\;(18)^{2n+1}} \tag{1.11}$$

$$R_{14} \equiv \frac{4}{\pi\sqrt{5}} = \frac{41}{72} - \frac{685}{5.72^3}\,\frac{1}{2}\,\frac{1.3}{4^2} + \frac{1329}{5^2.72^5}\,\frac{1.3}{2.4}\,\frac{1.3.5.7}{4^2.8^2} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(644n+41)\left(\frac{1}{2}\right)_n\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^3\;5n\;(72)^{2n+1}} \tag{1.12}$$

$$R_{15} \equiv \frac{4}{\pi} = \frac{1123}{882} - \frac{22583}{882^3}\,\frac{1}{2}\,\frac{1.3}{4^2} + \frac{44043}{882^5}\,\frac{1.3}{2.4}\,\frac{1.3.5.7}{4^2.8^2} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(21460n+1123)\left(\frac{1}{2}\right)_n\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^3\;(882)^{2n+1}} \tag{1.13}$$

$$R_{16} \equiv \frac{2\sqrt{3}}{\pi} = 1 + \frac{9}{9}\,\frac{1}{2}\,\frac{1.3}{4^2} + \frac{17}{9^2}\,\frac{1.3}{2.4}\,\frac{1.3.5.7}{4^2.8^2} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(8n+1)\left(\frac{1}{2}\right)_n\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^3\;9^n} \tag{1.14}$$

$$R_{17} \equiv \frac{1}{2\pi\sqrt{2}} = \frac{1}{9} + \frac{11}{9^3}\,\frac{1}{2}\,\frac{1.3}{4^2} + \frac{21}{9^5}\,\frac{1.3}{2.4}\,\frac{1.3.5.7}{4^2.8^2} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(10n+1)\left(\frac{1}{2}\right)_n\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^3\ (9)^{2n+1}} \tag{1.15}$$

$$R_{18} \equiv \frac{1}{3\pi\sqrt{3}} = \frac{3}{49} + \frac{43}{49^2}\ \frac{1}{2}\ \frac{1.3}{4^2} + \frac{83}{49^2}\ \frac{1.3}{2.4}\ \frac{1.3.5.7}{4^2.8^2} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(40n+3)\left(\frac{1}{2}\right)_n\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^3\ 49^{2n+1}} \tag{1.16}$$

$$R_{19} \equiv \frac{2}{\pi\sqrt{11}} = \frac{19}{99} + \frac{299}{99^3}\ \frac{1}{2}\ \frac{1.3}{4^2} + \frac{579}{99^5}\ \frac{1.3}{2.4}\ \frac{1.3.5.7}{4^2.8^2} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(280n+19)\left(\frac{1}{2}\right)_n\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^3\ 99^{2n+1}} \tag{1.17}$$

and

$$R_{20} \equiv \frac{1}{2\pi\sqrt{2}} = \frac{1103}{99^2} + \frac{27493}{99^6}\ \frac{1}{2}\ \frac{1.3}{4^2} + \frac{53883}{99^{10}}\ \frac{1.3}{2.4}\ \frac{1.3.5.7}{4^2.8^2} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(26390n+1103)\left(\frac{1}{2}\right)_n\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^3\ 99^{4n+2}} \tag{1.18}$$

Qureshi *et al.* derived the following formulas [7]:

$$ {}_4F_3\left(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{11}{10}; 1, 1, \frac{1}{10}; \frac{1}{81}\right) = \frac{9\sqrt{2}}{4\pi} \tag{1.19}$$

$$ {}_4F_3\left(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{43}{40}; 1, 1, \frac{3}{40}; \frac{1}{2401}\right) = \frac{49\sqrt{3}}{27\pi} \tag{1.20}$$

$$ {}_4F_3\left(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{299}{280}; 1, 1, \frac{19}{280}; \frac{1}{9801}\right) = \frac{198\sqrt{11}}{209\pi} \tag{1.21}$$

$$ {}_4F_3\left(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{27493}{26390}; 1, 1, \frac{1103}{26390}; \frac{1}{96059601}\right) = \frac{9801\sqrt{2}}{4412\pi} \tag{1.22}$$

and

$$ {}_4F_3\left(\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{558731543}{545140134}; 1, 1, \frac{13591409}{545140134}; -\frac{1728}{(640320)^3}\right) = \frac{(640320)^{\frac{3}{2}}}{12(13591409)\pi} \tag{1.23}$$

Furthermore, generalized hypergeometric function is defined as:

$$ {}_\alpha F_\beta \left[\begin{array}{lc} a_1, a_2, \cdots, a_\alpha \ ; & \\ & z \\ b_1, b_2, \cdots, b_\beta \ ; & \end{array} \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_\alpha)_k z^k}{(b_1)_k (b_2)_k \cdots (b_\beta)_k k!} \tag{1.24}$$

Meijer G-function is defined in the form of Line integral as [1]:

$$G^{m,n}_{p,q}\left(\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\middle| z\right) = \frac{1}{2\pi\iota}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j - s)} z^s \, ds \quad (1.25)$$

2. Main Results

In this section, we establish set of five definite integrals involving hypergeometric and Gamma functions:

Theorem 1. *Each of the following assertion holds true:*

$$\int_0^\infty e^{-t} \sqrt[10]{t}\ {}_3F_3\Big(\frac{1}{2},\frac{1}{4},\frac{3}{4};1,1,\frac{1}{10};\frac{t}{81}\Big)dt = \frac{9\sqrt{2}}{4\pi}\ \Gamma\Big(\frac{11}{10}\Big) \quad (2.1)$$

$$\int_0^\infty e^{-t} \sqrt[40]{t^3}\ {}_3F_3\Big(\frac{1}{2},\frac{1}{4},\frac{3}{4};1,1,\frac{3}{40};\frac{t}{2401}\Big)dt = \frac{49\sqrt{3}}{27\pi}\ \Gamma\Big(\frac{43}{40}\Big) \quad (2.2)$$

$$\int_0^\infty e^{-t} \sqrt[280]{t^{19}}\ {}_3F_3\Big(\frac{1}{2},\frac{1}{4},\frac{3}{4};1,1,\frac{19}{280};\frac{t}{9801}\Big)dt = \frac{198\sqrt{11}}{209\pi}\ \Gamma\Big(\frac{299}{280}\Big) \quad (2.3)$$

$$\int_0^\infty e^{-t} \sqrt[26390]{t^{1103}}\ {}_3F_3\Big(\frac{1}{2},\frac{1}{4},\frac{3}{4};1,1,\frac{1103}{26390};\frac{t}{96059601}\Big)dt$$

$$= \frac{9801\sqrt{2}}{4412\pi}\ \Gamma\Big(\frac{27493}{26390}\Big) \quad (2.4)$$

and

$$\int_0^\infty e^{-t} \sqrt[545140134]{t^{13591409}}\ {}_3F_3\Big(\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1,\frac{13591409}{545140134};\frac{t}{151931373056000}\Big)dt$$

$$= \frac{(640320)^{\frac{3}{2}}}{12(13591409)\pi}\ \Gamma\Big(\frac{558731543}{545140134}\Big) \quad (2.5)$$

provided that each member of the assertions (2.1) to (2.5) exists.

Proofs. We prove assertions (2.1) to (2.5), as given below:
We first prove our assertion (2.1) as:

$$\int_0^\infty e^{-t} \sqrt[10]{t}\ {}_3F_3\Big(\frac{1}{2},\frac{1}{4},\frac{3}{4};1,1,\frac{1}{10};\frac{t}{81}\Big)dt$$

$$= -\Gamma\Big(\frac{11}{10}\Big)\ \frac{\Big(\prod_{k=1}^{3}\ \Gamma(b_k)\Big)\sum_{k=0}^{\infty}\frac{(\frac{1}{81}-t)^k\ G^{3,2}_{4,4}\left(\begin{matrix} 1,\frac{1}{10}+k,1+k,1+k \\ \frac{1}{4}+k,\frac{1}{2}+k,\frac{3}{4}+k,\frac{11}{10}+k \end{matrix}\middle| -\frac{1}{t}\right)}{k!}}{\prod_{k=1}^{4}\ \Gamma(a_k)}$$

$$= \Gamma\Big(\frac{11}{10}\Big) \sum_{k=0}^{\infty} \frac{81^{-k}(\frac{1}{4})_k(\frac{1}{2})_k(\frac{3}{4})_k(\frac{11}{10})_k}{k!\ (\frac{1}{10})_k\big((1)_k\big)^2}$$

$$= \Gamma\Big(\frac{11}{10}\Big)\ {}_4F_3\Big(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{11}{10}; 1, 1, \frac{1}{10}; \frac{1}{81}\Big)$$

Now using (1.19) and after little algebra, we have

$$\int_0^{\infty} e^{-t}\ \sqrt[10]{t}\ {}_3F_3\Big(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}; 1, 1, \frac{1}{10}; \frac{t}{81}\Big)dt = \frac{9\sqrt{2}}{4\pi}\ \Gamma\Big(\frac{11}{10}\Big).$$

This completes our demonstration of the first assertion (2.1).
Next, we prove of second assertion (2.2), as:

$$\int_0^{\infty} e^{-t}\ \sqrt[40]{t^3}\ {}_3F_3\Big(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}; 1, 1, \frac{3}{40}; \frac{t}{2401}\Big)dt$$

$$= -\Gamma\Big(\frac{43}{40}\Big)\ \frac{\Big(\prod_{k=1}^{3}\ \Gamma(b_k)\Big) \sum\limits_{k=0}^{\infty} \frac{(\frac{1}{2401}-t)^k\ G_{4,4}^{3,2}\left(\left.\begin{matrix} 1, \frac{3}{40}+k, 1+k, 1+k \\ \frac{1}{4}+k, \frac{1}{2}+k, \frac{3}{4}+k, \frac{43}{40}+k \end{matrix}\right| -\frac{1}{t}\right)}{k!}}{\prod_{k=1}^{4}\ \Gamma(a_k)}$$

$$= \Gamma\Big(\frac{43}{40}\Big) \sum_{k=0}^{\infty} \frac{2401^{-k}(\frac{1}{4})_k(\frac{1}{2})_k(\frac{3}{4})_k(\frac{43}{40})_k}{k!\ (\frac{3}{40})_k\big((1)_k\big)^2}$$

$$= \Gamma\Big(\frac{43}{40}\Big)\ {}_4F_3\Big(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{43}{40}; 1, 1, \frac{3}{40}; \frac{1}{2401}\Big)$$

Now using (1.20) and applying little algebra, we have

$$\int_0^{\infty} e^{-t}\ \sqrt[40]{t^3}\ {}_3F_3\Big(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}; 1, 1, \frac{3}{40}; \frac{t}{2401}\Big)dt = \frac{49\sqrt{3}}{27\pi}\ \Gamma\Big(\frac{43}{40}\Big).$$

This completes our demonstration of the first assertion (2.2).
Next, we prove of second assertion (2.3), as:

$$\int_0^{\infty} e^{-t}\ \sqrt[280]{t^{19}}\ {}_3F_3\Big(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}; 1, 1, \frac{19}{280}; \frac{t}{9801}\Big)dt$$

$$= -\Gamma\Big(\frac{299}{280}\Big)\ \frac{\Big(\prod_{k=1}^{3}\ \Gamma(b_k)\Big) \sum\limits_{k=0}^{\infty} \frac{(\frac{1}{9801}-t)^k\ G_{4,4}^{3,2}\left(\left.\begin{matrix} 1, \frac{19}{280}+k, 1+k, 1+k \\ \frac{1}{4}+k, \frac{1}{2}+k, \frac{3}{4}+k, \frac{299}{280}+k \end{matrix}\right| -\frac{1}{t}\right)}{k!}}{\prod_{k=1}^{4}\ \Gamma(a_k)}$$

$$= \Gamma\Big(\frac{299}{280}\Big)\ \sum_{k=0}^{\infty} \frac{9801^{-k}(\frac{1}{4})_k(\frac{1}{2})_k(\frac{3}{4})_k(\frac{299}{280})_k}{k!\ (\frac{19}{280})_k\big((1)_k\big)^2}$$

$$= \Gamma\Big(\frac{299}{280}\Big)\ {}_4F_3\Big(\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{299}{280};1,1,\frac{19}{280};\frac{1}{9801}\Big)$$

Now using (1.21) and making numerical computation, we have

$$\int_0^{\infty} e^{-t}\ \sqrt[280]{t^{19}}\ {}_3F_3\Big(\frac{1}{2},\frac{1}{4},\frac{3}{4};1,1,\frac{19}{280};\frac{t}{9801}\Big)dt$$

$$= \Gamma\Big(\frac{299}{280}\Big)\ \frac{198\sqrt{11}}{209\pi} = \frac{198\sqrt{11}}{209\pi}\ \Gamma\Big(\frac{299}{280}\Big).$$

This completes our demonstration of the first assertion (2.3).
Next, we prove of second assertion (2.4), as:

$$\int_0^{\infty} e^{-t}\ \sqrt[26390]{t^{1103}}\ {}_3F_3\Big(\frac{1}{2},\frac{1}{4},\frac{3}{4};1,1,\frac{1103}{26390};\frac{t}{96059601}\Big)dt$$

$$= -\Gamma\Big(\frac{27493}{26390}\Big)\ \frac{\Big(\prod_{k=1}^{3}\ \Gamma(b_k)\Big)\sum_{k=0}^{\infty}\frac{(\frac{1}{96059601}-t)^k\ G_{4,4}^{3,2}\left(\left.\begin{matrix}1,\frac{1103}{26390}+k,1+k,1+k\\ \frac{1}{4}+k,\frac{1}{2}+k,\frac{3}{4}+k,\frac{27493}{26390}+k\end{matrix}\right| -\frac{1}{t}\right)}{k!}}{\prod_{k=1}^{4}\ \Gamma(a_k)}$$

$$= \Gamma\Big(\frac{27493}{26390}\Big)\ \sum_{k=0}^{\infty} \frac{96059601^{-k}(\frac{1}{4})_k(\frac{1}{2})_k(\frac{3}{4})_k(\frac{27493}{26390})_k}{k!\ (\frac{1103}{26390})_k\big((1)_k\big)^2}$$

$$= \Gamma\Big(\frac{27493}{26390}\Big)\ {}_4F_3\Big(\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{27493}{26390};1,1,\frac{1103}{26390};\frac{1}{96059601}\Big)$$

Now using (1.22) and by numerical computation, we have

$$= \frac{9801\sqrt{2}}{4412\pi}\ \Gamma\Big(\frac{27493}{26390}\Big).$$

This completes our demonstration of the first assertion (2.4).
Next, we prove assertion (2.5), as:

$$\int_0^{\infty} e^{-t}\ \sqrt[545140134]{t^{13591409}}\ {}_3F_3\Big(\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1,\frac{13591409}{545140134};\frac{t}{151931373056000}\Big)dt$$

$$= -\Gamma\Big(\frac{558731543}{545140134}\Big) \frac{\Big(\prod_{k=1}^{3}\ \Gamma(b_k)\Big) \sum\limits_{k=0}^{\infty} \frac{(-\frac{1}{151931373056000}-t)^k\ G_{4,4}^{3,2}\left(\begin{array}{c} 1,\frac{13591409}{545140134}+k,1+k,1+k \\ \frac{1}{6}+k,\frac{1}{2}+k,\frac{5}{6}+k,\frac{558731543}{545140134}+k \end{array}\middle| -\frac{1}{t}\right)}{k!}}{\prod_{k=1}^{4}\ \Gamma(a_k)}$$

$$= \Gamma\Big(\frac{558731543}{545140134}\Big) \sum_{k=0}^{\infty} \frac{\Big(-\frac{1}{151931373056000}\Big)^k (\frac{1}{6})_k(\frac{1}{2})_k(\frac{5}{6})_k(\frac{558731543}{545140134})_k}{k!\ (\frac{13591409}{545140134})_k\big((1)_k\big)^2} =$$

$$= \Gamma\Big(\frac{558731543}{545140134}\Big)\ {}_4F_3\Big(\frac{1}{6},\frac{1}{2},\frac{5}{6},\frac{558731543}{545140134};1,1,\frac{13591409}{545140134};-\frac{1728}{(640320)^3}\Big)$$

Now using (1.23) and applying little algebra, we obtain

$$\int_0^{\infty} e^{-t}\ \sqrt[545140134]{t^{13591409}}\ {}_3F_3\Big(\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1,\frac{13591409}{545140134};\frac{t}{151931373056000}\Big)dt$$

$$= \frac{(640320)^{\frac{3}{2}}}{12(13591409)\pi}\ \Gamma\Big(\frac{558731543}{545140134}\Big).$$

This completes our demonstration of the first assertion (2.5).
Hence we completed our proof of Theorem 1.

3. Conclusion

In our present investigation, we have made use of the Gamma function as well as the hypergeometric and the generalized hypergeometric functions with a view developing several definite integrals involving the elliptic integrals of the first and the second kind, respectively. The numerical approximation of these definite integrals and the corresponding hypergeometric functions are also presented. The results derived in this article are believed to be new and would extend and unify those that are available in the scientific literature.

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